

APPENDIX: Determining the Participation Fraction of Ions

We return to the second definition of Lorentz thrust force in Eq. (1), $F_{\parallel} = NeBv_d$. The number of ions between the electrodes may be estimated as $N = (\rho_{\text{NaCl}}/m_{\text{NaCl}})A_eL = (8.4 \pm 0.9) \times 10^{22}$, where $m_{\text{NaCl}} = \frac{1}{2}(m_{\text{Na}} + m_{\text{Cl}}) = 29.2m_p$. (Here $m_{\text{Na}} = 23.0m_p$, $m_{\text{Cl}} = 35.5m_p$, and m_p is the proton mass.)

Drift velocity v_d is usually expressed in terms of mobility u via $v_d = uE$, where u has units of (m/s)/(V/m) or m^2/Vs and $E = V/L$ in a uniform field like the one considered here. Mobility depends on concentration, and its value is typically extrapolated to the theoretical limit of “infinite dilution,” where interactions between the ions themselves can be neglected. For Na and Cl these limiting mobilities are $u_{0,\text{Na}} = 5.19 \times 10^{-8} \text{ m}^2/\text{Vs}$ and $u_{0,\text{Cl}} = 7.91 \times 10^{-8} \text{ m}^2/\text{Vs}$.²² We therefore define a mean limiting mobility $\bar{u}_0 = 6.6 \times 10^{-8} \text{ m}^2/\text{Vs}$.

To extrapolate this mean mobility to higher concentrations like our $c = 2.6 \text{ mol/L}$ requires a short excursion into electrochemistry. The nomenclature here is forbidding, but the story in brief is as follows. Mobility u is proportional to another quantity known as *molar conductivity* Λ , or conductivity per mole of solute. For strong electrolytes like NaCl, Λ is known to follow *Kohlrausch’s law*, $\Lambda(c) = \Lambda_0 - k\sqrt{c}$.²³ More directly observable, however, is conductivity itself, denoted by κ . (Conductivity is the inverse of resistivity ρ_e , with units of $\Omega^{-1}\text{m}^{-1}$.) And in practice, experimentalists measure the generalization of conductivity to AC currents known as *permittivity* ε , which for our purposes may be taken as proportional to κ . Dielectric relaxation spectroscopy reveals that the permittivity of NaCl solutions is well described over $0 \leq c \leq 4.55 \text{ mol/L}$ by $\varepsilon(c) = \delta_\varepsilon c - \beta_\varepsilon c^{3/2}$ with $\beta_\varepsilon = (4.1 \pm 0.2 \text{ L/mol})^{3/2}$ and $\delta_\varepsilon = 16.8 \pm 0.4 \text{ L/mol}$.²⁴ This agrees with Kohlrausch’s law, since $\varepsilon(c) \propto \kappa(c) \propto c\Lambda(c)$. Physically, it means that conductivity increases linearly with c at low concentrations, as might be expected. At higher concentrations, this linear dependence weakens as the ions begin to interact with each other. And at the highest concentrations, conductivity begins to *decrease* as oppositely charged ions pair up and no longer feel the electric field. Conductivity peaks when $d\varepsilon/dc = \delta_\varepsilon - (3/2)\beta_\varepsilon\sqrt{c} = 0$, or $c = (2\delta_\varepsilon/3\beta_\varepsilon)^2 = 7.5 \text{ mol/L}$, in agreement with recent experimental data.²⁵

The relevance of all this for our boats is that mobility u is proportional to Λ , and hence to $\varepsilon/c = \delta_\varepsilon - \beta_\varepsilon\sqrt{c}$, and goes over to \bar{u}_0 as $c \rightarrow 0$. We may therefore write

$$u(c) = \bar{u}_0(1 - \gamma\sqrt{c}), \quad (\text{A1})$$

where $\gamma = \beta_\varepsilon/\delta_\varepsilon = 0.24 \pm 0.01 \text{ (L/mol)}^{1/2}$. It follows that mobility at our concentration is reduced by some 40% from its limiting value: $u(2.6) = 0.61 \bar{u}_0 = (4.0 \pm 0.2) \times 10^{-8} \text{ m}^2/\text{Vs}$. Current, in other words, does not scale linearly with concentration. A more comprehensive analysis of the interplay between concentration, mobility, and applied voltage (including the configuration of the power supply) might reveal a “sweet spot” for optimal boat performance.

Table III. Mean ion drift velocity v_d with corresponding Lorentz force F_{\parallel} and current I_{th} , assuming all ions participate.

	v_d ($\mu\text{m/s}$)	F_{\parallel} (mN)	I_{th} (A)
9 V	5.3 ± 1.1	5.0 ± 1.6	3.6 ± 0.9
18 V	5.6 ± 2.6	5.3 ± 2.7	3.8 ± 1.8
27 V	4.0 ± 2.5	3.8 ± 2.5	2.7 ± 1.7

With the above value for mobility u , we obtain the drift velocities $v_d = uV/L$ and corresponding Lorentz forces $F_{\parallel} = NeBv_d$ listed in Table III, where the electrode potential comes from Ohm’s law $V = IR$. These forces differ by less than 10% from those in Table I. This level of agreement between two independent expressions for Lorentz force calls for physical explanation. One way to understand it is to define the theoretical current I_{th} that would be obtained if *all* the ions in the region between the electrodes participated: $I_{\text{th}} = Ne/t = Nev_d/L$ (where $t = L/v_d$ is the mean time required to cross from one electrode to the other). Numerical values of I_{th} are listed in Table III, and agree closely with the actual measured currents in Table I. Digging a little deeper, we define the *ion participation fraction* f_p as the ratio of the measured current I to its theoretical maximum value I_{th} . Combining the various definitions and expressions given above, we find (after much cancellation) the rather remarkable result:

$$f_p \equiv \frac{I}{I_{\text{th}}} = \frac{m_{\text{NaCl}}}{e\rho_e \rho_{\text{NaCl}} u} = 92 \pm 10\%. \quad (\text{A2})$$

Thus, the physical implication of the close numerical agreement between the results in Tables I and III is that nearly all the ions present are helping to push the boat.

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